Determine if each of the following converges or diverges.
If it converges, write "CONVERGES". If it diverges, write "DIVERGES".
Justify each answer using proper mathematical reasoning, algebra and/or calculus.

$$\sum_{n=1}^{\infty} \frac{n \ln n}{(n+1)^3}$$

SCORE: _____ / $9\frac{1}{2}$ PTS

$\frac{1}{2}$ $0 \le \frac{n \ln n}{(n+1)^3} < \frac{n \ln n}{n^3} = \frac{\ln n}{n^2}$ which is positive for $n \ge 2$, continuous for $n \ge 1$ $\frac{1}{2}$ $\frac{1}{2}$

and decreasing for $n \ge 2$ (since $\frac{d}{dx}\frac{\ln x}{x^2} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3} < 0$ for $x \ge \sqrt{e}$)

$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{N\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_2^N = \lim_{N\to\infty}\left(-\frac{\ln N}{N} - \frac{1}{N} + \frac{1}{2}\ln 2 + \frac{1}{2}\right) = -0 - 0 + \frac{1}{2}\ln 2 + \frac{1}{2} = \frac{1}{2}\ln 2 + \frac{1}{2}$ ie. converges

(since $\lim_{N\to\infty}\frac{\ln N}{N} = \lim_{N\to\infty}\frac{\frac{1}{N}}{1} = 0$)

So, $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ converges by Integral Test and $\sum_{n=1}^{\infty}\frac{n \ln n}{(n+1)^3}$ converges by Comparison Test

$$\sum_{n=1}^{\infty} \frac{\sin 2n}{n + n^2}$$

SCORE: _____ / $6\frac{1}{2}$ PTS

$\frac{1}{2}$ $0 < \left|\frac{\sin 2n}{n+n^2}\right| < \frac{1}{n+n^2} < \frac{1}{n^2}$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges ($p = 2 > 1$) by p-Series Test, so $\sum_{n=1}^{\infty}\left|\frac{\sin 2n}{n+n^2}\right|$ converges by Comparison Test

and $\sum_{n=1}^{\infty}\frac{\sin 2n}{n+n^2}$ converges by Absolute Convergence Test

Determine if each of the following converges or diverges.
If it converges, write "CONVERGES". If it diverges, write "DIVERGES".
Justify each answer using proper mathematical reasoning, algebra and/or calculus.

$$\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$$

SCORE: _____ / 3 PTS

Let $b_n = \frac{5^n}{4^n} = \left(\frac{5}{4}\right)^n$ $\frac{1}{2}$

$$\lim_{n\to\infty} \frac{\frac{5^n}{3^n+4^n}}{\frac{5^n}{4^n}} = \lim_{n\to\infty} \frac{5^n}{3^n+4^n}\,\frac{4^n}{5^n} = \lim_{n\to\infty} \frac{1}{(\frac{3}{4})^n + 1} = \frac{1}{0+1} = 1 \neq 0$$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{4}$

$\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n$ $\frac{1}{4}$ diverges ($|r| = \frac{5}{4} > 1$) $\frac{1}{4}$ by Geometric Series Test, $\frac{1}{4}$ so $\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$ diverges by Limit Comparison Test $\frac{1}{2}$

SEE ALTERNATE SOLUTIONS ON VERSION 2 KEY – GRADE AGAINST ONLY ONE SOLUTION

$$\sum_{n=1}^{\infty} (1 - \tfrac{1}{n})^{n^2}$$

SCORE: _____ / 6 PTS

$$\lim_{n\to\infty} \sqrt[n]{(1-\tfrac{1}{n})^{n^2}} = \lim_{n\to\infty} (1-\tfrac{1}{n})^n = e^{-1} < 1$$

(since $\lim\limits_{n\to\infty} \ln(1-\frac{1}{n})^n = \lim\limits_{n\to\infty} n\ln(1-\frac{1}{n}) = \lim\limits_{n\to\infty} \frac{\ln(1-\frac{1}{n})}{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{\frac{1}{1-\frac{1}{n}}(\frac{1}{n^2})}{-\frac{1}{n^2}} = \lim\limits_{n\to\infty} -\frac{1}{1-\frac{1}{n}} = -\frac{1}{1-0} = -1$)

So, $\sum_{n=1}^{\infty} (1-\frac{1}{n})^{n^2}$ converges by Root Test

$$\sum_{n=1}^{\infty} \frac{3^{2n}(n!)^2}{(2n-1)!}$$

SCORE: _____ / 5 PTS

$$\lim_{n\to\infty} \frac{3^{2(n+1)}((n+1)!)^2}{(2(n+1)-1)!}\,\frac{(2n-1)!}{3^{2n}(n!)^2} = \lim_{n\to\infty} \frac{3^{2n+2}(n+1)^2(n!)^2}{(2n+1)!}\,\frac{(2n-1)!}{3^{2n}(n!)^2} = \lim_{n\to\infty} \frac{3^2(n+1)^2}{(2n+1)(2n)} = \frac{9}{4} > 1$$

So, $\sum_{n=1}^{\infty} \frac{3^{2n}(n!)^2}{(2n-1)!}$ diverges by Ratio Test